Math 757, Fall 2015
Final Exam

Posted: Saturday, Dec 12 2015
Due: Monday, Dec 14 2015, 2:30-3:30 pm

The exam is due in my office at the above time,
either delivered into my hand or slipped underneath my door.
Answer all questions.
Show work for all problems.

Collaboration policy: In working on the problem sets, feel free to discuss the
problems with your classmates. However, no collaboration is allowed in writ-
ing up the solutions. Each student is to write up his or her own solution and is
expected to be able to explain and reproduce the work she or she submits.

NAME:

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1. (20 points)
Solve using the method of characteristics:
(a) \( u_x + x^2 u_y = -yu, \quad u = f(y) \) on \( x = 0 \).
(b) \( x_1 u_{x_1} + x_2 u_{x_2} = -x_1 x_2 \) in \( U = \{ x_1 > 0, x_2 > 0, x_1 x_2 > 2 \} \) and \( u = 1 \) on \( \partial U \).
   Hint: Derive and solve a second order ODE for the quantity
   \[ z(s) = u(x_1(s), x_2(s)) \].

2. (20 points)
   (a) Assume \( u \in \text{C}^2(\Omega) \cap \text{C}^0(\overline{\Omega}) \) satisfies the equation
   \[ \Delta u - 2 \cos(|\nabla u|^2) = 1 \].
   Show that \( u \) cannot attain a maximum at an interior point of \( \Omega \).
   (b) Assume \( u \in \text{C}^2(\Omega) \cap \text{C}^0(\overline{\Omega}), \Omega \) bounded, satisfies the equation
   \[ \Delta u = u^3 - u \] in \( \Omega \),
   and that \( u = 0 \) on \( \partial \Omega \). Show that \( -1 \leq u \leq 1 \) throughout \( \Omega \).

3. (20 points)
Consider the 1-dimensional heat equation
\[ u_t = u_{xx}, \quad (0.1) \]
on the unit interval \( x \in [0, 1] \). We impose homogeneous Dirichlet boundary conditions,
\[ u(t, 0) = u(t, 1) = 0, \quad (0.2) \]
and give initial data
\[ u(0, x) = u_0(x), \quad (0.3) \]
We’ll assume whatever smoothness we need of the solution.
(a) Multiply \( (0.1) \) by \( u \), integrate by parts in \( x \) over \([0, 1]\), use the boundary conditions, and finally integrate in \( t \) over \([0, T]\), to obtain the estimate
\[ \frac{1}{2} \int_0^1 u(T, x)^2 \, dx + \int_0^T \int_0^1 u_x^2(t, x) \, dx \, dt = \frac{1}{2} \int_0^1 u_0^2(x) \, dx. \]
(b) Instead of multiplying \( (0.1) \) by \( u \) try to multiply by \( f(u) \). Letting \( F \) be a primitive of \( f \) \( (F' = f) \) argue that
\[
\int_0^1 F(u(x,T)) \, dx = \int_0^1 F(u_0(x)) \, dx - \int_0^T \int_0^1 F''(u)u_x^2 \, dx \, dt,
\]
provided \( f(0) = 0 \).

(c) Argue that if \( F \) is a convex function whose derivative vanishes at zero, then
\[
\int_0^1 F(u(x,T)) \, dx \leq \int_0^1 F(u_0(x)) \, dx.
\]

4. (20 points)
Consider the heat equation
\[
v_t = v_{xx},
\] (0.4)
In the computations you are free to assume whatever smoothness is required to justify the calculations. First: Show that the equation
\[
u_t + c(t)u = u_{xx}
\]
can be reduced to (0.4) by the transform
\[
v(x,t) := u(x,t) \exp \left( \int_0^t c(s) \, ds \right).
\]
Then show that if \( v \) is any solution of (0.4) then so are the functions \( u(x,t) \) given by

(a) \( u(x,t) := \int_0^x v(y,t) \, dy \), provided that \( v_x(x_0, t) = 0 \) for all \( t > 0 \).

(b) \( u(x,t) := \int_0^t v(x,s) \, ds \), provided that \( v(x, t_0) = 0 \) for all \( x \).

(c) \( u(x,t) := \int_a^b v(x-y,t)\phi(y) \, dy \), for any constants \( a \) and \( b \) and smooth function \( \phi \).

(d) \( u(x,t) := \int_a^b v(x-t-s)\phi(s) \, ds \), for any constants \( a \) and \( b \) and smooth function \( \phi \).

(e) \( u(x,t) := \int_a^t v(x-t-s)\phi(s) \, ds \), provided \( v(x,0) = 0 \) for all \( x \).

(f) \( u(x,t) := v(\lambda x, \lambda^2 t) \), for any \( \lambda \in \mathbb{R} \).

(g) \( u(x,t) := xv_x(x,t) + 2tv_t(x,t) \).

5. (20 points)
Consider the wave equation with a zero’th order source/sink term:
\[
u_{tt} - u_{xx} + \lambda u = 0,
\] (0.5)
with initial data \( u(0, x) = g(x) \) and \( u_t(0, x) = h(x) \). Let’s assume for simplicity that the data have compact support and that \( u \in C^2 \). Verify that the energy
\[
E(t) := \frac{1}{2} \int_R u_t^2 + u_x^2 + \lambda u^2 \, dx
\]
is conserved.
Next, let’s also add a friction term and consider
\[
u_{tt} - u_{xx} + \alpha u_t + \lambda u = 0 .
\]
Make the ansatz that \( u(t, x) = w(t, x)e^{-\frac{\alpha}{2}t} \) and determine the equation \( w \) satisfies. In this case show that the same energy \( E(t) \) as above decreases provided \( \alpha > 0 \).